Betti numbers of Stanley–Reisner rings with pure resolutions

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Abstract

Let Δ be simplicial complex and let $k[\Delta]$ denote the Stanley–Reisner ring corresponding to Δ. Suppose that $k[\Delta]$ has a pure free resolution. Then we describe the Betti numbers and the Hilbert–Samuel multiplicity of $k[\Delta]$ in terms of the h–vector of Δ. As an application, we derive a linear equation system for the components of the h–vector of the clique complex of an arbitrary chordal graph.

1 Introduction

Let k denote an arbitrary field. Let R be the graded ring $k[x_1, \ldots, x_n]$. The vector space $R_s = k[x_1, \ldots, x_n]_s$ consists of the homogeneous polynomials of total degree s, together with 0.

In [9] R. Fröberg characterized the graphs G such that G has a linear free resolution. He proved:

Theorem 1.1 *Let G be a simple graph on n vertices. Then $R/I(G)$ has linear free resolution precisely when $\overline{G}$, the complementary graph of G is chordal.*

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In [6] E. Emtander generalized Theorem 1.1 for generalized chordal hypergraphs.

In this article we prove explicit formulas for the Betti numbers of the Stanley–Reisner ring $k[\Delta]$ such that $k[\Delta]$ has a pure free resolution in terms of the h–vector of Δ.

In Section 2 we collected some basic results about simplicial complices, free resolutions, Hilbert fuctions and Hilbert series. We present our main results in Section 3.

2 Preliminaries

2.1 Free resolutions

Recall that for every finitely generated graded module M over R we can associate to M a *minimal graded free resolution*

$$0 \longrightarrow \bigoplus_{i=1}^{\beta_p} R(-d_{p,i}) \longrightarrow \bigoplus_{i=1}^{\beta_{p-1}} R(-d_{p-1,i}) \longrightarrow \ldots \longrightarrow \bigoplus_{i=1}^{\beta_0} R(-d_{0,i}) \longrightarrow M \longrightarrow 0,$$

where $p \leq n$ and $R(-j)$ is the free R-module obtained by shifting the degrees of R by j.

Here the natural number β_k is the k'th *total Betti number* of M and p is the projective dimension of M.

The module M has a *pure resolution* if there are constants $d_0 < \ldots < d_p$ such that

$$d_{0,i} = d_0, \ldots, d_{p,i} = d_p$$

for all i. If in addition

$$d_i = d_0 + i,$$

for all $1 \leq i \leq p$, then we call the minimal free resolution to be d_0–*linear*.

In [20] Theorem 2.7 the following bound for the Betti numbers was proved.

Theorem 2.1 *Let M be an R–module having a pure resolution of type $(d_0, \ldots, d_p)$ and Betti numbers $\beta_0, \ldots, \beta_p$, where p is the projective dimension of M. Then*

$$\beta_i \geq \binom{p}{i} \tag{1}$$

for each $0 \leq i \leq p$.

2.2 Hilbert–Serre Theorem

Let $M = \bigoplus_{i \geq 0} M_i$ be a finitely generated nonnegatively graded module over the polynomial ring R. We call the formal power series

$$H_M(z) := \sum_{i=0}^{\infty} h_M(i) z^i$$

the *Hilbert–series* of the module M.

The Theorem of Hilbert–Serre states that there exists a (unique) polynomial $P_M(z) \in \mathbb{Q}[z]$, the so-called *Hilbert polynomial* of M, such that $h_M(i) = P_M(i)$ for each $i >> 0$. Moreover, P_M has degree dim $M - 1$ and (dim $M - 1$)! times the leading coefficient of P_M is the *Hilbert–Samuel multiplicity* of M, denoted here by $e(M)$.

Hence there exist integers $m_0, \ldots, m_{d-1}$ such that $h_M(z) = m_0 \cdot \binom{z}{d-1} + m_1 \cdot \binom{z}{d-2} + \ldots + m_{d-1}$, where $\binom{z}{r} = \frac{1}{r!} z(z-1)\ldots(z-r+1)$ and $d := \dim M$. Clearly $m_0 = e(M)$.

We can summarize the Hilbert-Serre theorem as follows:

Theorem 2.2 *(Hilbert–Serre) Let M be a finitely generated nonnegatively graded R–module of dimension d, then the following stetements hold:*
(a) There exists a (unique) polynomial $P(z) \in \mathbb{Z}[z]$ such that the Hilbert–series $H_M(z)$ of M may be written as

$$H_M(z) = \frac{P(z)}{(1-z)^d}$$

(b) d is the least integer for which $(1-z)^d H_M(z)$ is a polynomial.

2.3 Simplicial complices and Stanley–Reisner rings

We say that $\Delta \subseteq 2^{[n]}$ is a *simplicial complex* on the vertex set $[n] = \{1, 2, \ldots, n\}$, if Δ is a set of subsets of $[n]$ such that Δ is a down–set, that is, $G \in \Delta$ and $F \subseteq G$ implies that $F \in \Delta$, and $\{i\} \in \Delta$ for all i.

The elements of Δ are called *faces* and the *dimension* of a face is one less than its cardinality. An r-face is an abbreviation for an r-dimensional face. The dimension of Δ is the dimension of a maximal face. We use the notation $\dim(\Delta)$ for the dimension of Δ.

If $\dim(\Delta) = d - 1$, then the $(d+1)$–tuple $(f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ is called the *f-vector* of Δ, where $f_i(\Delta)$ denotes the number of i–dimensional faces of Δ.

Let Δ be an arbitrary simplicial complex on $[n]$. The *Stanley–Reisner ring* $k[\Delta] := R/I(\Delta)$ of Δ is the quotient of the ring R by the *Stanley–Reisner ideal*

$$I(\Delta) := \langle x^F : \ F \notin \Delta \rangle,$$

generated by the non–faces of Δ.

The following Theorem was proved in [1] Theorem 5.1.7.

Theorem 2.3 *Let Δ be a $d-1$–dimensional simplicial complex with f-vector $f(\Delta) := (f_{-1}, \ldots, f_{d-1})$. Then the Hilbert–series of the Stanley–Reisner ring $k[\Delta]$ is*

$$H_{k[\Delta]}(z) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

$\square$

Recall from Theorem 2.2 that a homogeneous k-algebra M of dimension d has a Hilbert series of the forn

$$H_M(z) = \frac{P(z)}{(1-z)^d}$$

where $P(z) \in \mathbb{Z}[z]$. Let Δ be a $(d-1)$–dimensional simplicial complex and write

$$H_{k[\Delta]}(z) = \frac{\sum_{i=0}^{d} h_i z^i}{(1-z)^d}. \qquad (2)$$

Lemma 2.4 *The f-vector and the h-vector of a $(d-1)$–dimensional simplicial complex Δ are related by*

$$\sum_i h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}.$$

In particular, the h-vector has length at most d, and

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}$$

for each $j = 0, \ldots, d$.

$\square$

3 Our main result

In the following Theorem we describe the Betti numbers of $k[\Delta]$ in terms of the h–vector of Δ.

Theorem 3.1 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a pure free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{3}$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{4}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then

$$\beta_i = \sum_{\ell=0}^{d_i} (-1)^{\ell+i+1} \binom{n-d}{\ell} h_{d_i-\ell}$$

for each $0 \leq i \leq p$.

Remark. Clearly $h_i = 0$ for each $i > d$.

Remark. J. Herzog and M. Kühl proved similar formulas for the Betti number in [16] Theorem 1. Here we did not assume that the Stanley–Reisner ring $k[\Delta]$ with pure resolution is Cohen–Macaulay.

Proof. Let $M := k[\Delta]$ denote the Stanley–Reisner ring of Δ. Then we infer from Theorem 2.3 that

$$H_M(z) = \frac{\sum_{i=0}^{d} h_i z^i}{(1-z)^d}. \tag{5}$$

Since the Hilbert–series is additive on short exact sequences, and since

$$H_R(z) = \frac{1}{(1-z)^n},$$

and consequently

$$H_{R(-s)}(z) = \frac{z^s}{(1-z)^n},$$

the pure resolution

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{6}$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow M \longrightarrow 0. \tag{7}$$

yields to

$$H_M(z) = \frac{1}{(1-z)^n} + \sum_{i=0}^{p} (-1)^{i+1} \beta_i \frac{z^{d_i}}{(1-z)^n}, \tag{8}$$

where $p = pdim(M)$.

Write $d := \dim M$, and let $m := \operatorname{codim}(M) = n - d$. It follows from the Auslander–Buchbaum formula that $m \leq p$.

Comparing the two expressions (8) and (5) for H_M, we find

$$(1-z)^m \left(\sum_{i=0}^{d} h_i z^i \right) = \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{d_i} + 1 \tag{9}$$

Using the binomial Theorem we get that

$$\left(\sum_{j=0}^{n-d} (-1)^j \binom{n-d}{j} z^j \right) \left(\sum_{i=0}^{d} h_i z^i \right) = \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{d_i} \tag{10}$$

Comparing the coefficients on the two sides of (10), we get the result.

□

Corollary 3.2 *Let Δ be a $(d-1)$–dimensional simplicial complex. Then*

$$e(k[\Delta]) = f_{d-1}.$$

Proof. It follows from [1] Proposition 4.1.9 and (2) that

$$e(k[\Delta]) = \left(\sum_{i=0}^{d} h_i z^i \right) |_{z=1} = \sum_{i=0}^{d} h_i = f_{d-1}.$$

Corollary 3.3 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has an t–linear free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-t-p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{11}$$

$$\longrightarrow R(-t-1)^{\beta_1} \longrightarrow R(-t)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{12}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then

$$\beta_i = \sum_{\ell=0}^{t+i} (-1)^{\ell+i+1} h_{t+i-\ell} \binom{n-d}{\ell}$$

for each $0 \leq i \leq p$.

Corollary 3.4 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has an t–linear free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-t-p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{13}$$

$$\longrightarrow R(-t-1)^{\beta_1} \longrightarrow R(-t)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{14}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

If $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ is the h-vector of the complex Δ, then

$$\sum_{\ell=0}^{j} (-1)^{\ell} h_{j-\ell} \binom{n-d}{\ell} = 0.$$

for each $j > p + t$.

Proof. Let

$$P(z) := 1 + \sum_{i=0}^{p} (-1)^{i+1} \beta_i z^{t+i}$$

Clearly $\deg(P) \leq p + t$. Comparing the coefficients of both side of (10), we get the result. □

Corollary 3.5 *Let G be an arbitrary chordal graph. Let $\Delta := \Delta(G)$ be the clique complex of G and $d := dim(\Delta) + 1$. Let $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$ denote the h-vector of the complex Δ. Let p be the projective dimension of the Stanley–Reisner ring $k[\Delta]$. Then*

$$\sum_{\ell=0}^{j} (-1)^{\ell} h_{j-\ell} \binom{n-d}{\ell} = 0$$

for each $j > p + 2$.

Proof. This follows easily from Theorem 1.1 and Corollary 3.4. □

Corollary 3.6 *Let Δ be a $(d-1)$–dimensional simplicial complex. Suppose that the Stanley–Reisner ring $k[\Delta]$ has a pure free resolution*

$$\mathcal{F}_\Delta : 0 \longrightarrow R(-d_p)^{\beta_p} \longrightarrow \ldots \longrightarrow \tag{15}$$

$$\longrightarrow R(-d_1)^{\beta_1} \longrightarrow R(-d_0)^{\beta_0} \longrightarrow R \longrightarrow k[\Delta] \longrightarrow 0. \tag{16}$$

Here p is the projective dimension of the Stanley–Reisner ring $k[\Delta]$.

Then

$$\sum_{\ell=0}^{d_i} (-1)^{\ell+i+1} \binom{n-d}{\ell} h_{d_i-\ell} \geq \binom{p}{i} \tag{17}$$

for each $0 \leq i \leq p$.

Proof. This follows easily from Theorem 2.1 and Theorem 3.1. □

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